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On the sum of some alternating series

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ABSTRACT

The scope of the paper is the presentation of a simple and original method of generating the sum of certain alternating series.

1. Introduction

Let $\gamma \in (0, \pi)$, $A, B \in \mathbb{R}$, $B < A$. It is easy to verify that the trigonometric Fourier series of the 2π-period function $f(x)$ defined as

$$f(x) = \begin{cases} A, & \pi - \gamma < |x| \le \pi, \\ \dfrac{1}{2}(A + B), & |x| = \pi - \gamma, \\ B, & |x| < \pi - \gamma, \end{cases}$$

has the following form:

$$f(x) = \frac{1}{\pi}\,(\gamma A + (\pi - \gamma) B) + \frac{2(B - A)}{\pi} \sum_{n=1}^{\infty} \frac{\sin((\pi - \gamma) n)}{n} \cos(n x).$$

Hence, in the sequel, for $x = \pi - \gamma$ we get the formula

$$\gamma - \frac{\pi}{2} = \sum_{n=1}^{\infty} \frac{1}{n} \sin(2n(\pi - \gamma)) \tag{1.1}$$

or

$$\frac{1}{2} - y = \sum_{n=1}^{\infty} \frac{\sin(2\pi n y)}{n\pi} \quad (0 < y < 1).$$

This formula is well recognized and has been used by several authors, especially in the theory of Fourier's trigonometric series (see for example [1]). In this paper, on the grounds of Eq. (1.1), we shall generate the values of sums of the alternating series $S_{r,n}$ defined in the following way

$$S_{r,n} := \sum_{k=0}^{\infty} \left(\frac{1}{k r + n} - \frac{1}{(k+1) r - n} \right)$$

for $r = 3, 4, \ldots$ and $n = 1, 2, \ldots, \left\lfloor \frac{r-1}{2} \right\rfloor$. Such an approach to the problem seems to be entirely original.

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In the next two chapters, we shall designate the values of $S_{r,n}$, depending on the fact, if r is a prime or a composite number. When r is a prime number, it is easy to provide a general formula, although this involves a certain algebraic problem, yet unresolved by the authors of this paper (see Problem 1). When r is a composite number, the case is not so promising, and only for $r = 6$ and $r = 9$ some interesting results were obtained.

It seems that apart from the discussed method deeply rooted in the theory of Fourier's series, there is only one original approach to summing a wide class of alternating series, i.e., Newton's technique [2], which will be discussed on an appropriately generalized example in the last chapter.

There are yet another publications worth mentioning, i.e. [3] (focused on a certain classic Pringsheim's result), as well as [4], where an attempt was made at generalizing the theory of real alternating series to the respective vector series, and also [5,6].

2. Prime r — special cases

2.1. $r = 5$

Setting in (1.1), $\gamma = \frac{4\pi}{5}$ and $\gamma = \frac{3\pi}{5}$, respectively, we find the following system of equations

$$\begin{cases} \frac{3}{10}\pi \csc\frac{\pi}{5} = 2\cos\frac{\pi}{5} S_{5,1} + S_{5,2}, \\ \frac{1}{10}\pi\csc\frac{\pi}{5} = S_{5,1} - 2\cos\frac{\pi}{5} S_{5,2}, \end{cases}$$

which easily implies the formulae

$$S_{5,1} = \frac{\pi}{10}\csc\frac{\pi}{5}\,\frac{1+6\cos\frac{\pi}{5}}{1+4\cos^2\frac{\pi}{5}} = \frac{\pi}{5}\cot\frac{\pi}{5} = \frac{\pi}{100}\sqrt{10}\left(\sqrt{5}+1\right)\sqrt{5+\sqrt{5}},$$

and

$$S_{5,2} = \frac{\pi}{10}\csc\frac{\pi}{5}\,\frac{3-2\cos\frac{\pi}{5}}{1+4\cos^2\frac{\pi}{5}} = \frac{\pi}{5}\cot\frac{2\pi}{5} = \frac{\pi}{100}\sqrt{10}\left(\sqrt{5}-1\right)\sqrt{5-\sqrt{5}}.$$

2.2. $r = 7$

Assuming, in (1.1), $\gamma = \frac{k\pi}{7}$, $k = 1, 2, 3$, respectively, we find the following system of equations:

$$\begin{cases} \frac{5\pi}{14} = \sin\left(\frac{2\pi}{7}\right) S_{7,1} + \sin\left(\frac{4\pi}{7}\right) S_{7,2} - \sin\left(\frac{8\pi}{7}\right) S_{7,3}, \\ \frac{3\pi}{14} = \sin\left(\frac{4\pi}{7}\right) S_{7,1} + \sin\left(\frac{8\pi}{7}\right) S_{7,2} - \sin\left(\frac{2\pi}{7}\right) S_{7,3}, \\ \frac{\pi}{14} = -\sin\left(\frac{8\pi}{7}\right) S_{7,1} - \sin\left(\frac{2\pi}{7}\right) S_{7,2} + \sin\left(\frac{4\pi}{7}\right) S_{7,3}, \end{cases}$$

which, by Cramer's rule, implies

$$S_{7,k} = \frac{D_{7,k}}{D_7}, \quad k = 1, 2, 3,$$

where

$$\begin{aligned} D_7 &:= \begin{vmatrix} \sin\frac{2\pi}{7} & \sin\frac{4\pi}{7} & -\sin\frac{8\pi}{7} \\ \sin\frac{4\pi}{7} & \sin\frac{8\pi}{7} & -\sin\frac{2\pi}{7} \\ -\sin\frac{8\pi}{7} & -\sin\frac{2\pi}{7} & \sin\frac{4\pi}{7} \end{vmatrix} \\ &= 3\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}\sin\frac{8\pi}{7} - \sin^3\frac{2\pi}{7} - \sin^3\frac{4\pi}{7} - \sin^3\frac{8\pi}{7}. \end{aligned}$$

From the identities:

$$x^3 + y^3 + z^3 - 3xyz = (x+y+z)\left((x+y+z)^2 - 3(xy+xz+yz)\right) \tag{2.1}$$

and (see [7]):

$$\left(x-2\,\sin\frac{2\pi}{7}\right)\left(x-2\,\sin\frac{4\pi}{7}\right)\left(x-2\,\sin\frac{8\pi}{7}\right)=x^3-\sqrt{7}\,x^2+\sqrt{7},\tag{2.2}$$

we obtain

$$D_7=-\frac{1}{8}\left(2\,\sin\frac{2\pi}{7}+2\,\sin\frac{4\pi}{7}+2\,\sin\frac{8\pi}{7}\right)^3=-\frac{7}{8}\sqrt{7}.$$

Moreover, since $\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}=-\frac{1}{2}$ the following formulae can be deduced:

$$D_{7,1}:=\frac{\pi}{14}\begin{vmatrix}5 & \sin\frac{4\pi}{7} & -\sin\frac{8\pi}{7}\\ 3 & \sin\frac{8\pi}{7} & -\sin\frac{2\pi}{7}\\ 1 & -\sin\frac{2\pi}{7} & \sin\frac{4\pi}{7}\end{vmatrix}=-\frac{\pi}{2}\left(\frac{3}{4}+\cos\frac{2\pi}{7}\right),$$

$$D_{7,2}:=\frac{\pi}{14}\begin{vmatrix}\sin\frac{2\pi}{7} & 5 & -\sin\frac{8\pi}{7}\\ \sin\frac{4\pi}{7} & 3 & -\sin\frac{2\pi}{7}\\ -\sin\frac{8\pi}{7} & 1 & \sin\frac{4\pi}{7}\end{vmatrix}=-\frac{\pi}{2}\left(\frac{3}{4}+\cos\frac{4\pi}{7}\right),$$

$$D_{7,3}:=\frac{\pi}{14}\begin{vmatrix}\sin\frac{2\pi}{7} & \sin\frac{4\pi}{7} & 5\\ \sin\frac{4\pi}{7} & \sin\frac{8\pi}{7} & 3\\ -\sin\frac{8\pi}{7} & -\sin\frac{2\pi}{7} & 1\end{vmatrix}=\frac{\pi}{2}\left(\frac{3}{4}+\cos\frac{8\pi}{7}\right).$$

Hence, we obtain

$$S_{7,k}=(-1)^{\frac{(k+2)(3k+1)}{2}}\,\frac{4\sqrt{7}}{49}\,\pi\,\left(\frac{3}{4}+\cos\left(\frac{2\,k\,\pi}{7}\right)\right),\quad k=1,2,3.\tag{2.3}$$

2.3. $r=11$

Using the formulae presented in [8], the following equations may be generated:

$$S_{11,1}=\omega\left(\frac{1}{4}-2\,\cos\left(\frac{10\,\pi}{11}\right)-\cos\left(\frac{8\,\pi}{11}\right)\right),\tag{2.4}$$

$$S_{11,2}=\omega\left(-\frac{1}{4}+2\,\cos\left(\frac{2\,\pi}{11}\right)+\cos\left(\frac{6\,\pi}{11}\right)\right),\tag{2.5}$$

$$S_{11,3}=\omega\left(\frac{1}{4}-2\,\cos\left(\frac{8\,\pi}{11}\right)-\cos\left(\frac{2\,\pi}{11}\right)\right),\tag{2.6}$$

$$S_{11,4}=\omega\left(\frac{1}{4}-2\,\cos\left(\frac{4\,\pi}{11}\right)-\cos\left(\frac{10\,\pi}{11}\right)\right),\tag{2.7}$$

$$S_{11,5}=\omega\left(\frac{1}{4}-2\,\cos\left(\frac{6\,\pi}{11}\right)-\cos\left(\frac{4\,\pi}{11}\right)\right),\tag{2.8}$$

where $\omega=\frac{4\sqrt{11}}{121}\,\pi$.

3. Prime *r* — the general case

Let $p\in\mathbb{N}$ be a prime number, $p\geq 5$. Put $r=p$. Setting, in (1.1), successively, $\gamma=\frac{k\pi}{p}$, $k=1,2,\ldots,\lfloor p/2\rfloor$, we find the following system of equations:

$$\frac{p-2k}{2p}\,\pi = \sum_{n=1}^{\lfloor p/2\rfloor} \sin\left(\frac{2kn}{p}\,\pi\right) S_{p,n}$$

for $k = 1, 2, \ldots, \lfloor p/2\rfloor$, which have, by Cramer's rule, the following solution:

$$S_{p,n} = \frac{D_{p,n}}{D_p},$$

where

$$D_{p,n} := \frac{\pi}{2p}\begin{vmatrix} \sin\frac{2\pi}{p} & \sin\frac{4\pi}{p} & \ldots & \sin\frac{2(n-1)\pi}{p} & p-2 & \sin\frac{2(n+1)\pi}{p} & \ldots & \sin\frac{2\lfloor p/2\rfloor\pi}{p} \\ \sin\frac{4\pi}{p} & \sin\frac{8\pi}{p} & \ldots & \sin\frac{4(n-1)\pi}{p} & p-4 & \sin\frac{4(n+1)\pi}{p} & \ldots & \sin\frac{4\lfloor p/2\rfloor\pi}{p} \\ \sin\frac{6\pi}{p} & \sin\frac{12\pi}{p} & \ldots & \sin\frac{6(n-1)\pi}{p} & p-6 & \sin\frac{6(n+1)\pi}{p} & \ldots & \sin\frac{6\lfloor p/2\rfloor\pi}{p} \\ \vdots & & & & & & & \\ \sin\frac{2\lfloor p/2\rfloor\pi}{p} & \sin\frac{4\lfloor p/2\rfloor\pi}{p} & \ldots & \sin\frac{2(n-1)\lfloor p/2\rfloor\pi}{p} & 1 & \sin\frac{2(n+1)\lfloor p/2\rfloor\pi}{p} & \ldots & \sin\frac{2\lfloor p/2\rfloor^2\pi}{p} \end{vmatrix},$$

for every $1 \le n \le \lfloor p/2\rfloor$ and

$$D_p = \det\left[\sin\left(\frac{2kn}{p}\,\pi\right)\right]_{\lfloor p/2\rfloor\times\lfloor p/2\rfloor}.$$

It was verified by numerical calculations for primes $p \le 1051$ that the following equality holds:

$$D_p = \begin{cases} \left(-\frac{p}{4}\right)^{(p-1)/4} & \text{when } p \equiv 1 (\mathrm{mod}\ 4), \\ (-1)^{(p-3)/4}\left(\frac{p}{4}\right)^{(p-1)/4} & \text{when } p \equiv 3 (\mathrm{mod}\ 4). \end{cases} \tag{3.1}$$

Remark 3.1. A case of expanded determinant $\det[\sin(k\,x_n)]_{p\times p}$, where $x_1, \ldots, x_n \in \mathbb{C}$, is known (see, for example, problem 10849 from Amer. Math. Monthly [9] or problem 350 in [10]).

Problem 1. Is Eq. (3.1) true for all prime numbers?

4. Composite *r* — special cases

4.1. $r = 4$

From (1.1) for $\gamma = \frac{3\pi}{4}$ the classic Leibnitz formula can be obtained (see also [11–13] and especially [14]):

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

4.2. $r = 6$

From (1.1) for $\gamma = \frac{\pi}{6}$ and $\gamma = \frac{\pi}{3}$ the following system of equations could be generated:

$$\begin{cases} \frac{2}{9}\sqrt{3}\,\pi = S_{6,1} + \frac{1}{2}\,S_{3,1}, \\ \frac{1}{9}\sqrt{3}\,\pi = S_{6,1} - \frac{1}{2}\,S_{3,1}, \end{cases}$$

which implies

$$S_{3,1} = \frac{\sqrt{3}}{9}\,\pi \quad \text{and} \quad S_{6,1} = \frac{\sqrt{3}}{6}\,\pi, \tag{4.1}$$

both are compatible with [15].

4.3. $r = 9$

From (1.1) for $\gamma = \frac{\pi}{9}, \frac{2\pi}{9}$ and $\frac{4\pi}{9}$, respectively, the following system of equations can be obtained:

$$\begin{cases} \dfrac{7}{18}\pi = \left(\sin\dfrac{2\pi}{9}\right) S_{9,1} + \left(\sin\dfrac{4\pi}{9}\right) S_{9,2} + \left(\sin\dfrac{8\pi}{9}\right) S_{9,4} + \dfrac{\sqrt{3}}{6} S_{3,1}, \\ \dfrac{5}{18}\pi = \left(\sin\dfrac{4\pi}{9}\right) S_{9,1} + \left(\sin\dfrac{8\pi}{9}\right) S_{9,2} - \left(\sin\dfrac{2\pi}{9}\right) S_{9,4} - \dfrac{\sqrt{3}}{6} S_{3,1}, \\ \dfrac{1}{18}\pi = \left(\sin\dfrac{8\pi}{9}\right) S_{9,1} - \left(\sin\dfrac{2\pi}{9}\right) S_{9,2} - \left(\sin\dfrac{4\pi}{9}\right) S_{9,4} + \dfrac{\sqrt{3}}{6} S_{3,1}, \end{cases}$$

which, by (4.1) implies

$$\begin{cases} \dfrac{\pi}{3} = \left(\sin\dfrac{2\pi}{9}\right) S_{9,1} + \left(\sin\dfrac{4\pi}{9}\right) S_{9,2} + \left(\sin\dfrac{8\pi}{9}\right) S_{9,4}, \\ \dfrac{\pi}{3} = \left(\sin\dfrac{4\pi}{9}\right) S_{9,1} + \left(\sin\dfrac{8\pi}{9}\right) S_{9,2} - \left(\sin\dfrac{2\pi}{9}\right) S_{9,4}, \\ 0 = -\left(\sin\dfrac{8\pi}{9}\right) S_{9,1} + \left(\sin\dfrac{2\pi}{9}\right) S_{9,2} + \left(\sin\dfrac{4\pi}{9}\right) S_{9,4}. \end{cases} \tag{4.2}$$

However, much to our surprise, by (2.1) we have

$$\begin{vmatrix} \sin\dfrac{2\pi}{9} & \sin\dfrac{4\pi}{9} & \sin\dfrac{8\pi}{9} \\ \sin\dfrac{4\pi}{9} & \sin\dfrac{8\pi}{9} & -\sin\dfrac{2\pi}{9} \\ -\sin\dfrac{8\pi}{9} & \sin\dfrac{2\pi}{9} & \sin\dfrac{4\pi}{9} \end{vmatrix} = 3\sin\frac{2\pi}{9}\sin\frac{4\pi}{9}\sin\frac{8\pi}{9} + \sin^3\frac{2\pi}{9} - \sin^3\frac{4\pi}{9} + \sin^3\frac{8\pi}{9} = 0$$

since (for example [16,17]):

$$\left(x - 2\sin\frac{2\pi}{9}\right)\left(x + 2\sin\frac{4\pi}{9}\right)\left(x - 2\sin\frac{8\pi}{9}\right) = x^3 - 3x + \sqrt{3}. \tag{4.3}$$

So (4.2) can be reduced to the following form:

$$\begin{cases} \dfrac{\pi}{3} - \left(\sin\dfrac{8\pi}{9}\right) S_{9,4} = \left(\sin\dfrac{2\pi}{9}\right) S_{9,1} + \left(\sin\dfrac{4\pi}{9}\right) S_{9,2}, \\ \dfrac{\pi}{3} + \left(\sin\dfrac{2\pi}{9}\right) S_{9,4} = \left(\sin\dfrac{4\pi}{9}\right) S_{9,1} + \left(\sin\dfrac{8\pi}{9}\right) S_{9,2}. \end{cases}$$

Hence, we obtain only:

$$S_{9,1} - S_{9,4} = \frac{4}{9}\pi\left(\cos\frac{\pi}{18} - \sin\frac{\pi}{9}\right),$$

and

$$S_{9,2} + S_{9,4} = \frac{4}{9}\pi\left(\cos\frac{\pi}{18} - \sin\frac{2\pi}{9}\right).$$

4.4. $r = 8$

Adding the sides of (1.1) for $\gamma = \frac{\pi}{8}$ and $\gamma = \frac{3\pi}{8}$, we obtain the formula:

$$\sqrt{2}\,\frac{\pi}{4} = S_{8,1} + S_{8,3},$$

which is compatible with detailed Prudnikov's formulae $S_{8,1} = \frac{\pi}{48}\left(\sqrt{2}+1\right)$ and $S_{8,3} = \frac{\pi}{16}\left(\sqrt{2}-1\right)$ (see [15]).

4.5. $r = 16$

Conversely, adding the sides of (1.1) for $\gamma = \frac{15\pi}{16}$ and $\gamma = \frac{9\pi}{16}$, in the next step, (1.1) for $\gamma = \frac{13\pi}{16}$ and $\gamma = \frac{11\pi}{16}$, we obtain the following system of equations:

$$\begin{cases} \dfrac{\pi}{4} = \sin\dfrac{\pi}{8}\ (S_{16,1}+S_{16,7}) + \sin\dfrac{3\pi}{8}\ (S_{16,3}+S_{16,5}), \\ \dfrac{\pi}{4} = \sin\dfrac{3\pi}{8}\ (S_{16,1}+S_{16,7}) - \sin\dfrac{\pi}{8}\ (S_{16,3}+S_{16,5}). \end{cases}$$

Hence, the following form is derived:

$$S_{16,1}+S_{16,7} = \frac{\pi}{4}\left(\cos\frac{\pi}{8}+\sin\frac{\pi}{8}\right),$$

and

$$S_{16,3}+S_{16,5} = \frac{\pi}{4}\left(\cos\frac{\pi}{8}-\sin\frac{\pi}{8}\right).$$

5. Newton's method

The method is used for calculating the sums of series

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{s\,k+r}, \quad 0<r<s, \ \ r,s\in\mathbb{N}.$$

The acting of this method is presented by some special case in monograph [2] (also see the remarks in [18]). Due to a general nature of the method, it shall be discussed in more detail in this paper. It should be mentioned that identity (5.2) generated below is different from the corresponding Prudnikov's identity [15]. The advantages of Eq. (5.2) are easily verifiable special cases. Let us assume that $\varepsilon := \exp(i2\pi/(2n+1))$. Accordingly, we have

$$\sqrt[2n+1]{1} = \left\{\varepsilon^{\pm k}:\ \ k=0,1,\ldots,n\right\}.$$

In view of $(a\neq 0)$, we deduce the following decomposition:

$$\begin{aligned} \frac{(2\,n+1)\,a^{2n+r}}{x^r\,(x^{2n+1}-a^{2n+1})} &= \frac{1}{x-a} - \frac{2\,n+1}{a}\left(\sum_{\substack{0\le l\le\lfloor r/(2n+1)\rfloor \\ l(2n+1)<r}}\left(\frac{a}{x}\right)^{r-l(2n+1)}\right) + \sum_{k=1}^{n}\left(\frac{\varepsilon^{-k(2n+r)}}{x-a\,\varepsilon^{k}} + \frac{\varepsilon^{k(2n+r)}}{x-a\,\varepsilon^{-k}}\right) \\ &= \frac{1}{x-a} - \frac{2\,n+1}{a}\left(\sum_{\substack{0\le l\le\lfloor r/(2n+1)\rfloor \\ l(2n+1)<r}}\left(\frac{a}{x}\right)^{r-l(2n+1)}\right) + 2\sum_{k=1}^{n}\frac{x\,\cos\left(\frac{2k(r-1)}{2n+1}\,\pi\right) - a\,\cos\left(\frac{2kr}{2n+1}\,\pi\right)}{x^2-2\,a\,x\,\cos\left(\frac{2k\pi}{2n+1}\right)+a^2}. \end{aligned}$$

Next, we calculate (the cases: $r\equiv 1(\mathrm{mod}\ (2n+1))$ and all other $r\in\mathbb{N}$ should be discussed separately here):

$$\begin{aligned} \int_{-\infty}^{-1}\frac{2\,n+1}{x^r\,(x^{2n+1}-1)}\,\mathrm{d}x &= \ln 2 + \sum_{k=1}^{n}\left[2\,\cos\left(\frac{2\,k\,(r-1)\,\pi}{2\,n+1}\right)\ln\left(2\,\cos\left(\frac{k\,\pi}{2\,n+1}\right)\right)\right. \\ &\quad \left. + \frac{2\,k\,\pi}{2\,n+1}\,\sin\left(\frac{2\,k\,(r-1)\,\pi}{2\,n+1}\right)\right] + (2\,n+1)\,(-1)^r\sum_{\substack{0\le l\le\lfloor r/(2n+1)\rfloor \\ l(2n+1)<r-1}}\frac{(-1)^l}{r-1-l\,(2\,n+1)}. \end{aligned} \tag{5.1}$$

On the other hand, by expanding the integrand into Laurent's series, we obtain

$$\begin{aligned} \int_{-\infty}^{-1}\frac{2\,n+1}{x^r\,(x^{2n+1}-1)}\,\mathrm{d}x &= (2\,n+1)\int_{-\infty}^{-1}\left(\sum_{k=1}^{\infty}x^{-k(2n+1)-r}\right)\mathrm{d}x \\ &= (2\,n+1)\,(-1)^r\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k\,(2\,n+1)+r-1}, \end{aligned}$$

which by (5.1), after rescaling $r := r-1$ implies the identity

$$\begin{aligned} (2\,n+1)\,(-1)^{r-1}\sum_{k=T_{r,n}}^{\infty}\frac{(-1)^k}{k\,(2\,n+1)+r} &= \ln 2 + 2\sum_{k=1}^{n}\left[\cos\left(\frac{2\,k\,r\,\pi}{2\,n+1}\right)\ln\left(2\,\cos\left(\frac{k\,\pi}{2\,n+1}\right)\right)\right. \\ &\quad \left. + \frac{k\,\pi}{2\,n+1}\,\sin\left(\frac{2\,k\,r\,\pi}{2\,n+1}\right)\right], \end{aligned} \tag{5.2}$$

where

$$T_{r,n} := \min\left\{k\in\mathbb{Z}:\ -\lfloor(r+1)/(2n+1)\rfloor\le k \quad\text{and}\quad -r<k(2n+1)\right\}.$$

Hence, for $n = 1$ and $r = \pm 1$ we get the known formulae [18,15]:

$$\sum_{k=(1-r)/2}^{\infty} \frac{(-1)^{k-(1-r)/2}}{3k+r} = \frac{1}{3}\left(\frac{\pi}{\sqrt{3}} + r\,\ln 2\right).$$

However, for $n = 0$ and $r = 1$ we obtain (see also [19]):

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \ln 2.$$

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